

Scaling functions in the square Ising model

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Abstract.

We show and give the linear differential operators \mathcal{L}_q^{scal} of order $q = n^2/4 + n + 7/8 + (-1)^n/8$, for the integrals $I_n(r)$ which appear in the two-point correlation scaling function of Ising model $F_{\pm}(r) = \lim_{scaling} \mathcal{M}_{\pm}^{-2} < \sigma_{0,0} \sigma_{M,N} > = \sum_n I_n(r)$. The integrals $I_n(r)$ are given in expansion around $r = 0$ in the basis of the formal solutions of \mathcal{L}_q^{scal} with transcendental combination coefficients. We find that the expression $r^{1/4} \exp(r^2/8)$ is a solution of the Painlevé VI equation in the scaling limit. Combinations of the (analytic at $r = 0$) solutions of \mathcal{L}_q^{scal} sum to $\exp(r^2/8)$. We show that the expression $r^{1/4} \exp(r^2/8)$ is the scaling limit of the correlation function $C(N, N)$ and $C(N, N + 1)$. The differential Galois groups of the factors occurring in the operators \mathcal{L}_q^{scal} are given.

Key-words: Scaling functions of Ising model, diagonal correlation functions, diagonal form factors expansion, next-to-diagonal form factors, Painlevé VI equation, multidimensional integrals, modified Bessel functions, symplectic differential Galois group, orthogonal differential Galois group.

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1. Introduction

The scaling functions of the two-point correlation function of the square lattice Ising model $F_{\pm}(r)$ have been obtained by Wu et al. [1]. These scaling functions $F_{\pm}(r)$ are solutions of a *Painlevé* like equation [1, 2]. Symmetrical forms of these scaling functions have been also obtained by Palmer and Tracy [3, 4]

$$F_{\pm}(r) = \sum_n I_n, \quad (1)$$

where the I_n 's are n -dimensional integrals.

The expressions of I_1 and I_2 are known in closed form. Note that the integrand in the integrals I_n (see (11) below) is not algebraic in the variables but it is *holonomic*. Thus the integrals I_n must be solution of linear differential equations[‡]. These linear differential equations, annihilating the integrals I_n , are the main subject of this paper.

[‡] The integrals of a holonomic integrand are also holonomic.

The paper is organized as follows. Section 2 contains recalls on the scaling function of the two-point correlation function $F_{\pm}(r)$ and its symmetrical forms. Section 3 is a recall on the $f_N^{(n)}$, namely the *form factors* expansion of the diagonal correlation functions $C(N, N)$ on the square lattice. The linear differential operators \mathcal{L}_q^{scal} of order $q = n^2/4 + n + 7/8 + (-1)^n/8$, for these form factors at scaling have *no direct sum* decomposition. The general solutions of $f_N^{(1)}$, $f_N^{(2)}$ and $f_N^{(3)}$, at scaling, are given. Once the observation that the scaling limits of $f_N^{(1)}$ and $f_N^{(2)}$ are identical to I_1 and I_2 , we show in section 4, that the integrals I_n are solutions of the linear differential operators \mathcal{L}_q^{scal} . The proof is carried out by numerical methods, allowing to write the integrals I_n as an expansion of formal solutions of \mathcal{L}_q^{scal} , where the combination coefficients are *transcendental numbers*. Section 5 deals with the *sigma form* of Painlevé VI equation that annihilates $C(N, N)$, as well as its scaling limit. We seek, and find, four solutions to the scaled Painlevé equation. To each solution, we identify the corresponding solution of the N -dependent sigma form of Painlevé VI equation. In Section 6, we show that $x^{1/4} \exp(x^2/32)$ is the scaling limit of $C(N, N)$, and, in section 7, we show that it is *also the scaling limit of $C(N, N+1)$* . We show, in section 8, that the factors of the linear differential operators for the $f_N^{(n)}$ (as well as the corresponding operators in the scaling limit) have “special” differential Galois groups. Our conclusions are given in section 9.

2. Recalls on the scaling functions of the Ising model

The scaling functions are defined as [1] (where $\xi \cdot r = \sqrt{M^2 + N^2}$)

$$F_{\pm}(r) = \lim_{scaling} \mathcal{M}_{\pm}^{-2} \cdot \langle \sigma_{0,0} \sigma_{M,N} \rangle, \quad (2)$$

with $\mathcal{M}_{\pm} = (1 - t)^{1/8}$, where t is defined in section 3.

The scaling functions obtained in [1] are, for $T < T_c$

$$F_{-}(r) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{\pi^{2n}} g_{2n}(r)\right), \quad (3)$$

with

$$g_{2n}(r) = \frac{(-1)^n}{n} \int_1^{\infty} dy_1 \cdots \int_1^{\infty} dy_{2n} \cdot \prod_{j=1}^{2n} \frac{\exp(-ry_j)}{(y_j^2 - 1)^{1/2}(y_j + y_{j+1})} \cdot \prod_{j=1}^n (y_{2j}^2 - 1),$$

and for $T > T_c$

$$F_{+}(r) = X(r) \cdot F_{-}(r), \quad (4)$$

where

$$X(r) = \sum_{n=0}^{\infty} \frac{1}{\pi^{2n+1}} \cdot g_{2n+1}(r), \quad (5)$$

with

$$\begin{aligned} g_{2n+1}(r) &= (-1)^n \int_1^{\infty} dy_1 \cdots \int_1^{\infty} dy_{2n+1} \cdot \prod_{j=1}^{2n+1} \frac{\exp(-ry_j)}{(y_j^2 - 1)^{1/2}} \\ &\times \prod_{j=1}^{2n} \frac{1}{y_j + y_{j+1}} \cdot \prod_{j=1}^n (y_{2j}^2 - 1). \end{aligned} \quad (6)$$

It has been shown [1, 2] that the scaling functions F_{\pm} are remarkably given by *nonlinear equations of Painlevé type*:

$$F_{\pm}(x) = \left(\frac{\sinh(\psi(r)/2)}{\cosh(\psi(r)/2)} \right) \cdot \exp \frac{1}{4} \int_r^{\infty} (\sinh(\psi)^2 - (\frac{d\psi}{ds})^2) \cdot s \, ds, \quad (7)$$

where $\psi(r)$ verifies:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) - \frac{1}{2} \sinh(2\psi) = 0. \quad (8)$$

Setting

$$\zeta(r) = r \frac{d}{dr} \ln(F_{\pm}), \quad (9)$$

the equation (7) becomes:

$$(r \zeta'')^2 = 4 \cdot (r \zeta' - \zeta)^2 - 4 \cdot (\zeta')^2 \cdot (r \zeta' - \zeta) + (\zeta')^2. \quad (10)$$

The scaling functions are also given in a symmetrical form in [3] (see also [4]).

$$F_+(r) = \sum_{n=0}^{\infty} I_{2n+1}(r), \quad F_-(r) = 1 + \sum_{n=1}^{\infty} I_{2n}(r), \quad (11)$$

$$I_n = \frac{1}{n!} \int_0^{\infty} \frac{du_1}{2\pi} \cdots \int_0^{\infty} \frac{du_n}{2\pi} \prod_{i < j} \frac{(u_i - u_j)^2}{(u_i + u_j)^2} \prod_{i=1}^n \frac{1}{u_i} \exp\left(-\frac{r}{2}(u_i + 1/u_i)\right). \quad (12)$$

A direct computation gives

$$I_1 = \frac{1}{\pi} \cdot K_0(r). \quad (13)$$

A "less" direct computation yields

$$I_2 = \frac{1}{\pi^2} \cdot \left(\left(\frac{1}{2} - r^2 \right) \cdot K_0(r)^2 - r \cdot K_0(r) \cdot K_1(r) + r^2 \cdot K_1(r)^2 \right), \quad (14)$$

where K_0 (resp. K_1) is the (resp. derivative of the) *modified Bessel function* (see below).

3. The linear ODE of the form factors and their scaling limit

The diagonal correlation functions $C(N, N)$ of the square Ising model have a form factor expansion [5]

$$C(N, N) = (1-t)^{1/4} \cdot \left(1 + \sum_{n=1}^{\infty} f_N^{(2n)} \right), \quad T < T_c \quad (15)$$

with $t = \left(\sinh(2E^v/k_B T) \sinh(2E^h/k_B T) \right)^{-2}$, and

$$C(N, N) = (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} f_N^{(2n+1)}, \quad T > T_c \quad (16)$$

with $t = \left((\sinh(2E^v/k_B T) \sinh(2E^h/k_B T)) \right)^2$, where E^h and $E^v = E^h$ are the horizontal and vertical interaction energies of the Ising model.

The diagonal correlation functions $C(N, N)$ can be calculated from Toeplitz determinants [6, 7, 8]. They are also solutions of Painlevé VI *in its sigma form* [9].

The diagonal correlation functions $C(N, N)$, as well as the form factors $f_N^{(n)}$, write as *polynomials* in the complete elliptic integrals (see Appendix A for some recalls).

The diagonal form factors $f_N^{(n)}$ are n -dimensional integrals [5] and are annihilated by linear ODEs whose corresponding linear differential operators factorize, with factors such that the $f_N^{(n)}$ are “embedded” in the form factors $f_N^{(n+2k)}$

$$\left(\prod_{k=1}^n \bullet_{L_{2k}} \right) (f_N^{(2n-1)}) = 0, \quad \left(\prod_{k=0}^n \bullet_{L_{2k+1}} \right) (f_N^{(2n)}) = 0, \quad (17)$$

which means, for instance, that $f_N^{(1)}$ and $f_N^{(3)}$ are solutions of the linear ODEs:

$$L_2 f_N^{(1)} = 0, \quad L_4 \cdot L_2 f_N^{(3)} = 0. \quad (18)$$

The expressions of these order- n linear differential operators L_n have been obtained [5] for generic values of N (they are given up to $n = 10$ in [5]). This way, the scaling limit of these linear differential operators has been possible. The scaling limit amounts to taking both the limits $t \rightarrow 1$ and $N \rightarrow \infty$ in the linear differential operators. This is performed with the change of variable $x = (1 - t) \cdot N$, keeping the leading order of N .

In the scaling limit, the linear differential operators L_n in the variable t become linear differential operators L_n^{scal} in the scaling variable x , and we have shown [5] that the factors L_n^{scal} solve as polynomial expressions of *modified Bessel functions* of homogeneous degree. For some purposes in the sequel and easy references, we recall the factors L_1^{scal} , L_2^{scal} , L_3^{scal} , L_4^{scal} , L_5^{scal} and L_6^{scal} in Appendix B.

Call $B_0(x/2)$ and $K_0(x/2)$ the (respectively analytical at $x = 0$, and logarithmic) solutions of the *modified Bessel* differential operator (with D_x the derivative $\P d/dx$):

$$D_x^2 + \frac{1}{x} \cdot D_x - \frac{1}{4}. \quad (19)$$

We call $B_1(x/2)$ and $K_1(x/2)$ the first derivative of, respectively, $2B_0(x/2)$ and $-2K_0(x/2)$.

Consider the linear differential operator $L_4 \cdot L_2$ that annihilates the form factors $f^{(1)}(N)$ and $f^{(3)}(N)$, and denote by $L_4^{scal} \cdot L_2^{scal}$ the corresponding linear differential operators in the scaling limit.

The general solution of L_2^{scal} reads (omitting the argument $x/2$)

$$\text{sol}(L_2^{scal}) = c_1 \cdot B_0 + c_2 \cdot K_0. \quad (20)$$

The general solution of L_4^{scal} reads

$$\begin{aligned} \text{sol}(L_4^{scal}) = & c_3 \cdot \left(B_0^3 - x \cdot B_0^2 \cdot B_1 + B_0 \cdot B_1^2 + x \cdot B_1^3 \right) \\ & + c_4 \cdot \left(K_0^3 + x \cdot K_0^2 \cdot K_1 + K_0 \cdot K_1^2 - x \cdot K_1^3 \right) \\ & + c_5 \cdot \left(B_0^2 \cdot (3K_0 + x K_1) + B_1^2 \cdot (K_0 - 3x K_1) - 2B_0 B_1 \cdot (x K_0 + K_1) \right) \\ & + c_6 \cdot \left(K_0^2 \cdot (3B_0 - x B_1) + K_1^2 \cdot (B_0 + 3x \cdot B_1) - 2K_0 K_1 \cdot (B_1 - x B_0) \right), \end{aligned}$$

\P Similarly, we will also use, in this paper, the notations D_t for d/dt and D_s for d/ds .

and $L_4^{scal} \cdot L_2^{scal}$ solves as

$$\begin{aligned} \text{sol}(L_4^{scal} \cdot L_2^{scal}) &= \text{sol}(L_2^{scal}) \\ &\quad - B_0 \cdot \int K_0 \cdot \text{sol}(L_4^{scal}) \cdot x dx + K_0 \cdot \int B_0 \cdot \text{sol}(L_4^{scal}) \cdot x dx, \end{aligned}$$

i.e. the scaling limit of $f^{(1)}(N) + f^{(3)}(N)$ is *not a polynomial expression of modified Bessel functions*.

Similarly, for $T < T_c$, consider the linear differential operator $L_3 \cdot L_1$, with the constant and $f^{(2)}(N)$ as solutions. In the scaling limit the operator $L_3 \cdot L_1$ becomes $L_3^{scal} \cdot L_1^{scal}$ (with $L_1^{scal} = D_x$), and its general solution reads:

$$\begin{aligned} \text{sol}(L_3^{scal} \cdot D_x) &= c_0 + c_1 \cdot \left((2 - x^2) \cdot B_0^2 + 2x \cdot B_0 \cdot B_1 + x^2 \cdot B_1^2 \right) \\ &\quad + c_2 \cdot \left((2 - x^2) \cdot K_0^2 - 2x \cdot K_0 \cdot K_1 + x^2 \cdot K_1^2 \right) \\ &\quad + c_3 \cdot \left((x \cdot B_0 \cdot K_1 + x \cdot B_1 \cdot K_0 - x^2 \cdot B_1 \cdot K_1 - (x^2 - 2) \cdot B_0 \cdot K_0) \right). \end{aligned} \quad (21)$$

Note that $L_3^{scal} \cdot D_x$ has a direct sum decomposition (see Appendix B), but the operators (in the scaling limit) of higher order have not.

4. Linear differential equations of the I_n integrals (12)

If we compare I_1 given in (13) with (20), and I_2 given in (14) with (21), one remarks that the integrals are, respectively, solution of the linear differential operator L_2^{scal} and $L_3^{scal} \cdot D_x$, once the correspondence $r \rightarrow x/2$ has been made.

We now argue that the integrals $I_n(x)$ are solutions of the linear differential operator

$$\mathcal{L}_q^{scal} = L_{n+1}^{scal} \cdot L_{n-1}^{scal} \cdots L_2^{scal}, \quad q = (n+2)^2/4, \quad (22)$$

for n odd, and

$$\mathcal{L}_q^{scal} = L_{n+1}^{scal} \cdot L_{n-1}^{scal} \cdots L_1^{scal}, \quad q = (n+1)(n+3)/4, \quad (23)$$

for n even.

This will be proved numerically for the first $I_n(x)$, i.e. we show that:

$$\begin{aligned} L_2^{scal} \cdot I_1(x) &= 0, & (L_3^{scal} \cdot L_1^{scal}) \cdot I_2(x) &= 0, \\ (L_4^{scal} \cdot L_2^{scal}) \cdot I_3(x) &= 0, & (L_5^{scal} \cdot L_3^{scal} \cdot L_1^{scal}) \cdot I_4(x) &= 0. \end{aligned} \quad (24)$$

Let us show the method for the integrals $I_1(x)$ and $I_2(x)$ which are known in closed form expressions.

4.1. The integrals $I_1(x)$ and $I_2(x)$

With the formal solutions of L_2^{scal} at $x = 0$

$$S_1^{(1)} = S_2^{(1)} \cdot \ln(x) - \left(\frac{x^2}{16} + \frac{3x^4}{2048} + \cdots \right), \quad S_2^{(1)} = 1 + \frac{x^2}{16} + \frac{x^4}{1024} + \cdots,$$

we form the generic combination $c_1^{(1)} S_1^{(1)} + c_2^{(1)} S_2^{(1)}$ that we evaluate numerically (and its first derivative) at a fixed value of $x = x_0$. The integral $I_1(x)$ (and its first

derivative) are performed numerically for the same value of x . Solving the system

$$\begin{aligned} (c_1^{(1)} \cdot S_1^{(1)} + c_2^{(1)} \cdot S_2^{(1)})|_{x=x_0} &= I_1(x)|_{x=x_0}, \\ \frac{d}{dx} (c_1^{(1)} \cdot S_1^{(1)} + c_2^{(1)} \cdot S_2^{(1)})|_{x=x_0} &= \frac{d}{dx} I_1(x)|_{x=x_0}, \end{aligned} \quad (25)$$

in the constants $c_1^{(1)}$ and $c_2^{(1)}$, one obtains

$$c_1^{(1)} = -0.31830, \quad c_2^{(1)} = 0.25753, \quad (26)$$

which are easy to recognize, since $I_1(x)$ is known (and given in (13) with $r = x/2$), as

$$c_1^{(1)} = -\frac{1}{\pi}, \quad c_2^{(1)} = \frac{1}{\pi} \cdot (2 \ln(2) - \gamma).$$

where γ is Euler's constant.

The same calculations are performed for $L_3^{scal} \cdot D_x$ with the formal solutions written as

$$\begin{aligned} S_1^{(2)} &= S_3^{(2)} \cdot \ln(x)^2 + \left(\frac{5x^2}{8} + \frac{9x^4}{1024} + \frac{29}{221184} x^6 + \dots \right) \cdot \ln(x) \\ &\quad - \left(\frac{3x^2}{4} + \frac{x^4}{128} + \frac{19}{147456} x^6 + \dots \right), \\ S_2^{(2)} &= S_3^{(2)} \cdot \ln(x) + \left(\frac{5x^2}{16} + \frac{9x^4}{2048} + \frac{29}{442368} x^6 + \dots \right), \\ S_3^{(2)} &= 1 - \frac{x^2}{8} - \frac{x^4}{512} - \frac{x^6}{36864} + \dots, \quad S_4^{(2)} = 1. \end{aligned} \quad (27)$$

Similarly, the combination $c_1^{(2)} \cdot S_1^{(2)} + c_2^{(2)} \cdot S_2^{(2)} + c_3^{(2)} \cdot S_3^{(2)} + c_4^{(2)} \cdot S_4^{(2)}$, and its first three derivatives are evaluated numerically at a fixed value of $x = x_0$, and matched to the integral $I_2(x)$ (and its first three derivatives) performed numerically. Solving in the constants $c_j^{(2)}$, one obtains:

$$c_1^{(2)} = 0.0506605, \quad c_2^{(2)} = 0.0193443, \quad c_3^{(2)} = 0.052507, \quad c_4^{(2)} = 10^{-8}.$$

Here also, since $I_2(x)$ is known (and given in (14) with $r = x/2$), the constants are easy to recognize

$$\begin{aligned} c_1^{(2)} &= \frac{1}{2\pi^2}, \quad c_2^{(2)} = \frac{1}{\pi^2} \cdot (1 - 2 \ln(2) + \gamma), \\ c_3^{(2)} &= \frac{1}{2\pi^2} \cdot (1 - 2 \ln(2) + \gamma)^2 + \frac{1}{2\pi^2}, \quad c_4^{(2)} = 0. \end{aligned} \quad (28)$$

4.2. The integrals $I_3(x)$ and $I_4(x)$

Now, we consider the integral I_3 which should be a solution of $L_4^{scal} \cdot L_2^{scal}$, whose local exponents at $x = 0$ are 0, 0, 0, 0, 2, 2 (that we note $0^4, 2^2$). The formal solutions are written as:

$$\begin{aligned} S_1^{(3)} &= S_4^{(3)} \cdot \ln(x)^3 + \left(3 - \frac{21x^2}{8} - \frac{87x^4}{2048} + \dots \right) \cdot \ln(x)^2 \\ &\quad + \left(\frac{9}{2} + \frac{9x^2}{128} + \frac{81x^4}{8192} + \dots \right) \cdot \ln(x) + \left(3 + \frac{3x^2}{64} - \frac{75x^4}{2048} + \dots \right), \end{aligned}$$

† One may also, obviously, compute the combination of solutions and $I_1(x)$, i.e. (25), at two values of x

$$\begin{aligned}
S_2^{(3)} &= S_4^{(3)} \cdot \ln(x)^2 + \left(2 + \frac{x^2}{32} - \frac{x^4}{2048} + \dots\right) \cdot \ln(x) + \left(\frac{3}{2} + \frac{3x^2}{128} + \frac{27x^4}{8192} + \dots\right), \\
S_3^{(3)} &= S_4^{(3)} \cdot \ln(x) + \left(1 + \frac{x^2}{64} - \frac{x^4}{4096} + \dots\right), \quad S_4^{(3)} = 1 + \frac{7x^2}{16} + \frac{7x^4}{1024} + \dots, \\
S_5^{(3)} &= S_1^{(1)}, \quad S_6^{(3)} = S_2^{(1)}. \tag{29}
\end{aligned}$$

Similar calculations are performed, namely evaluating numerically the linear combination $\sum_j c_j^{(3)} S_j^{(3)}$ (and its five derivatives) matching with the integral $I_3(x)$ (and its five derivatives) at a given value of $x = x_0$. Solving in the constants $c_j^{(3)}$, one obtains:

$$\begin{aligned}
c_1^{(3)} &= -0.0322515/3!, & c_2^{(3)} &= -0.0184725/3!, & c_3^{(3)} &= -0.5789545/3!, \\
c_4^{(3)} &= 0.65939377/3!, & c_5^{(3)} &= 0.49900435/3!, & c_6^{(3)} &= -0.1942198/3!
\end{aligned}$$

The constant $c_1^{(3)}$ is easily recognized as $-\frac{1}{6\pi^3}$ and we may guess the constant $c_2^{(3)}$ as $(-1 + 2\ln(2) - \gamma)/(2\pi^3)$, but we have not attempted to recognize the other constants, because the number of correct digits is rather low. Note however, that if we evaluate, again, $I_3(x) - \sum_j c_j^{(3)} S_j^{(3)}$ with the obtained constants $c_j^{(3)}$ and for other values of x_0 , one obtains zero with the working accuracy.

Similar calculations are done for $I_4(x)$ with the basis of solutions at $x = 0$ of $L_5^{scal} \cdot L_3^{scal} \cdot L_1^{scal}$ (whose local exponents at $x = 0$ are $0^5, 2^3, 6$):

$$\begin{aligned}
S_1^{(4)} &= S_5^{(4)} \cdot \ln(x)^4 + \left(\frac{20}{3} + \frac{143}{12}x^2 + \frac{283}{1536}x^4 + \frac{5}{1024}x^6 + \dots\right) \cdot \ln(x)^3 \\
&\quad + \left(\frac{64}{3} + \frac{1765}{192}x^2 + \frac{1933}{12288}x^4 - \frac{275}{294912}x^6 + \dots\right) \cdot \ln(x)^2 \\
&\quad + \left(\frac{334}{9} + \frac{5771}{1152}x^2 - \frac{3829}{73728}x^4 - \frac{6509}{884736}x^6 + \dots\right) \cdot \ln(x) \\
&\quad + \frac{1549}{54} + \frac{21505}{13824}x^2 + \frac{466273}{884736}x^4 + \frac{102762373}{12740198400}x^6 + \dots, \tag{30} \\
S_2^{(4)} &= S_5^{(4)} \cdot \ln(x)^3 + \left(5 - \frac{223}{256}x^2 - \frac{247}{16384}x^4 + \frac{15}{131072}x^6 + \dots\right) \cdot \ln(x)^2 \\
&\quad + \left(\frac{32}{3} - \frac{473}{1536}x^2 + \frac{199}{98304}x^4 + \frac{659}{1179648}x^6 + \dots\right) \cdot \ln(x) \\
&\quad + \frac{167}{18} + \frac{485}{18432}x^2 - \frac{37915}{1179648}x^4 - \frac{8508439}{16986931200}x^6 + \dots, \\
S_3^{(4)} &= S_5^{(4)} \cdot \ln(x)^2 + \left(\frac{10}{3} - \frac{223}{384}x^2 - \frac{247}{24576}x^4 + \frac{5}{65536}x^6 + \dots\right) \cdot \ln(x) \\
&\quad + \frac{32}{9} - \frac{473}{4608}x^2 + \frac{199}{294912}x^4 + \frac{103027}{4246732800}x^6 + \dots, \\
S_4^{(4)} &= S_5^{(4)} \cdot \ln(x) + \frac{5}{3} - \frac{223}{768}x^2 - \frac{247}{49152}x^4 - \frac{30931}{707788800}x^6 + \dots, \\
S_5^{(4)} &= 1 - \frac{5x^2}{4} - \frac{5x^4}{256} - \frac{x^6}{2304} + \dots, \\
S_6^{(4)} &= S_1^{(2)}, \quad S_7^{(4)} = S_2^{(2)}, \quad S_8^{(4)} = S_3^{(2)}, \quad S_9^{(4)} = S_4^{(2)}.
\end{aligned}$$

The coefficients combination read

$$\begin{aligned}
c_1^{(4)} &= 0.0102659/4!, & c_2^{(4)} &= 0.0215279/4!, & c_3^{(4)} &= 0.423376/4!, \\
c_4^{(4)} &= -1.086613/4!, & c_5^{(4)} &= 1.063659/4!, & c_6^{(4)} &= -0.35704/4!, \\
c_7^{(4)} &= -0.02156/4!, & c_8^{(4)} &= 1.05496/4!, & c_9^{(4)} &= -1.38534/4!
\end{aligned}$$

Here also, the same numeric values of $c_j^{(4)}$ are obtained for *any other value* of x_0 .

Let us remark that if one just wants to check that $I_n(x)$ is a solution of \mathcal{L}_q^{scal} , one may proceed as follows. Call $\mathcal{I}_n(x, u)$ the integrand of $I_n(x)$ and integrate numerically

$$\mathcal{L}_q^{scal}(\mathcal{I}_n(x, u)), \quad (31)$$

for fixed n and various values of x , to get zero with the desired accuracy.

We claim that this continues for the higher I_n , and conclude that the integrals $I_n(x)$ are solutions of \mathcal{L}_q^{scal} , the scaling limit of the linear differential operator annihilating the form factors $f_N^{(n)}$.

4.3. The expansion around $x = 0$ of the integrals $I_n(x)$

The integrals $I_n(x)$ write as linear combination of all the formal solutions at $x = 0$ of \mathcal{L}_q^{scal}

$$I_n(x) = \sum_{j=1}^q c_j^{(n)} \cdot S_j^{(n)}, \quad (32)$$

Note that the numerical values $c_j^{(n)}$ do depend on the basis chosen for the formal solutions $S_j^{(n)}$ (see Appendix C which gives the constants for $I_2(x)$ with another combination of formal solutions). However, as an expansion, $I_n(x)$ is obviously *not* dependent on the basis. For instance, if we trust the guessed constants $c_1^{(3)}$ and $c_2^{(3)}$, the integral $I_3(x)$ reads:

$$\begin{aligned} I_3(x) = & -\frac{1}{6\pi^3} \cdot \left(1 + \frac{7}{16} \cdot x^2 + \frac{7}{1024} \cdot x^4 + \dots\right) \cdot \ln(x)^3 \\ & + \frac{1}{6\pi^3} \cdot \left(3 \cdot (2 \ln(2) - \gamma - 2) + \frac{21}{16} \cdot (1 + 2 \ln(2) - \gamma) \cdot x^2 \right. \\ & \quad \left. + \frac{3}{2048} \cdot (15 + 28 \ln(2) - 14 \gamma) \cdot x^4 + \dots\right) \cdot \ln(x)^2 \\ & - \frac{1}{6\pi^3} \cdot \left(8.124485 + 6.974855 x^2 + 0.117211 x^4 + \dots\right) \cdot \ln(x) \\ & + \frac{1}{6\pi^3} \cdot \left(-7.387058 + 7.260657 x^2 + 0.150333 x^4 + \dots\right). \end{aligned} \quad (33)$$

In front of $\ln(x)^3$, there is the overall constant $c_1^{(3)}$. For $\ln(x)^2$, there is no overall constant, because the series in front of $\ln(x)^2$ is a sum of two series with the combination coefficients $c_1^{(3)}$ and $c_2^{(3)}$. The same occurs for the others series in front of $\ln(x)$ and $\ln(x)^0$.

For $I_4(x)$, the expansion reads:

$$\begin{aligned} I_4(x) = & \frac{1}{24\pi^4} \cdot \left(1 - \frac{5}{4} x^2 - \frac{5}{256} x^4 - \frac{1}{2304} x^6 + \dots\right) \cdot \ln(x)^4 \\ & + \frac{1}{24\pi^4} \cdot (8.76369 + 9.29539 x^2 + 0.143287 x^4 + 0.00397265 x^6 + \dots) \cdot \ln(x)^3 \\ & + \frac{1}{24\pi^4} \cdot (38.2792 - 39.8374 x^2 - 0.611859 x^4 - 0.0176486 x^6 + \dots) \cdot \ln(x)^2 \\ & + \frac{1}{24\pi^4} \cdot (89.0014 + 91.2468 x^2 + 1.30355 x^4 + 0.0383979 x^6 + \dots) \cdot \ln(x) \\ & + \frac{1}{24\pi^4} \cdot (89.7926 - 88.8183 x^2 - 0.942513 x^4 - 0.0307719 x^6 + \dots). \end{aligned} \quad (34)$$

Remark 1: In the numerical evaluation of the constants $c_j^{(n)}$ by linear systems like (25), the issue of the numerical accuracy raises. For the left hand side (25) it is straightforward to have the series $S_j^{(n)}$ to any length. The difficulty is in the numerical evaluation of the multiple integrals (12) which controls the number of digits of the constants $c_j^{(n)}$.

Remark 2: In the evaluation of the linear systems like (25), the matching point $x = x_0$ is used. The value of x_0 can be any positive number, since the integrals $I_n(x)$ are defined for the positive $r = x/2$, and since the solutions $S_j^{(n)}$ are given by linear ODEs which have only $x = 0$ and $x = \infty$ as singularities.

We now turn to the diagonal correlation functions $C(N, N)$, which write as expansion on the form factors $f_N^{(n)}$. The linear differential equations that annihilate the $C(N, N)$ are of order $N + 1$. Appendix D shows that we can find the coefficients for generic N , but to go further, a recursion on these coefficients should be found. This seems hard to achieve. Fortunatly, there is a way to produce the linear differential equation at scaling that should contain the scaling limit of $C(N, N)$.

5. Painlevé VI sigma form equation in the scaling limit

It is known that the diagonal correlation functions of the Ising model, $C_N = C(N, N)$ verify the Painlevé VI equation *in its sigma form* [9]

$$\begin{aligned} & \left(t \cdot (t-1) \cdot \frac{d^2\sigma}{dt^2} \right)^2 + 4 \cdot \left((t-1) \frac{d\sigma}{dt} - \sigma - \frac{1}{4} \right) \cdot \left(t \cdot \frac{d\sigma}{dt} - \sigma \right) \cdot \frac{d\sigma}{dt} \\ &= N^2 \cdot \left((t-1) \cdot \frac{d\sigma}{dt} - \sigma \right)^2, \end{aligned} \quad (35)$$

where:

$$\sigma = t \cdot (t-1) \cdot \frac{d}{dt} \ln(C_N) - \frac{t}{4}, \quad T < T_c \quad (36)$$

$$\sigma = t \cdot (t-1) \cdot \frac{d}{dt} \ln(C_N) - \frac{1}{4}, \quad T > T_c \quad (37)$$

The scaling limit of this equation has been given by Jimbo and Miwa [9]. It is obtained by simply performing the appropriate change of variable, which amounts to changing to the variable $x = (1-t) \cdot N$ in *Painlevé VI sigma form*, keeping the leading N term. This gives the scaling equation (irrespective of the regime $T < T_c$ or $T > T_c$)

$$\begin{aligned} & x^2 \cdot \left(\frac{d^2\mu}{dx^2} \right)^2 + \left(4x \cdot \left(\frac{d\mu}{dx} \right) - x^2 - 4\mu \right) \cdot \left(\frac{d\mu}{dx} \right)^2 - \frac{1}{2} x \cdot \left(\frac{d\mu}{dx} \right) \cdot (1 - 4\mu) \\ &= \frac{1}{16} (1 - 4\mu)^2, \end{aligned} \quad (38)$$

with:

$$\mu = x \frac{d}{dx} \ln(C_{scal}(x)). \quad (39)$$

To make the expressions closer to the sigma form, one may define

$$\nu = x \cdot \frac{d}{dx} \ln(C_{scal}(x)) - \frac{1}{4}. \quad (40)$$

Equation (38) becomes

$$x^2 \cdot \left(\frac{d^2\nu}{dx^2} \right)^2 + 4 \cdot \left(x \cdot \left(\frac{d\nu}{dx} \right) - \nu - \frac{1}{4} \right) \cdot \left(\frac{d\nu}{dx} \right)^2 = \left(x \cdot \left(\frac{d\nu}{dx} \right) - \nu \right)^2. \quad (41)$$

Remark 3: The form (41) is equation (38) in [9], and identifies with (10) on ζ with $r = x/2$. It seems that this identification between (10) and (41) (and thus eq.(38) in [9]) has not been remarked.

Remark 4: Recall that (41) is for $C(N, N)$ while (10) is for $C(M, N)$. The factor \mathcal{M}_{\pm}^2 in (2) is taken care of by the $-1/4$ appearing in (40). Equation (41), which is the scaling limit of the equation that annihilates $C(N, N)$, (i.e. (35)) could also be the scaling limit of a *non-linear* equation (of the Painlevé type) for $C(M, N)$, generalizing (35), if it exists.

5.1. Some solutions of the Painlevé VI sigma form equation in the scaling limit

In order to find some of the (non logarithmic) solution $C_{scal}(x)$, we plug in (41) the form

$$C_{scal}(x) = x^\alpha \cdot \sum a_k \cdot x^k, \quad (42)$$

and solves, term by term, on the coefficients a_k . For generic α , one obtains

$$C_{scal}^{(I)}(x) = x^\alpha \cdot \exp\left(\pm i \frac{4\alpha - 1}{8\sqrt{\alpha}} \cdot x\right). \quad (43)$$

The value $\alpha = 1/4$ pops out as particular. When fixed and plugging (42) in (41), one obtains a one-parameter solution that reads:

$$\begin{aligned} C_{scal}(x) = & x^{1/4} \cdot \left(a_0 + a_2 \cdot x^2 + \frac{a_2}{64} \cdot x^4 + \frac{a_2 \cdot (a_0 - 8a_2)}{4608 a_0} \cdot x^6 \right. \\ & + \frac{a_2 \cdot (5a_0 - 64a_2)}{2359296 a_0} \cdot x^8 + \frac{a_2 \cdot (7a_0 - 104a_2)}{471859200 a_0} \cdot x^{10} \\ & \left. + \frac{a_2 \cdot (21a_0^2 - 296a_0a_2 - 512a_2^2)}{271790899200 a_0^2} \cdot x^{12} + \dots \right). \end{aligned} \quad (44)$$

For the value $a_0 = 0$, the solution corresponds to (43) for $\alpha = 1/4 + 2$. For $a_2 = 0$, the solution is

$$C_{scal}^{(II)}(x) = x^{1/4}. \quad (45)$$

Now we want to find whether there are particular values of a_2/a_0 for which the series $C_{scal}(x)/x^{1/4}$ in (44) verifies a linear ODE. For this we use the methods developped in [10, 11, 12, 13] (see also section 6 in [14], section 3 in [15]) and consider the series (44) modulo a given prime p_r . This way, as far as the coefficient a_2/a_0 is rational, its value is restricted to the interval $[1, p_r]$. We then let a_2/a_0 varies over the whole interval $[1, p_r]$ until a linear ODE is found. We have written the linear ODE in the (homogeneous) derivative $x \cdot D_x$, the coefficient-polynomials being of degree D , and searched for an ODE of order $Q \leq 6$, with $(Q+1)(D+1) \leq 220$. In this range of Q and D , there are only the values $a_2/a_0 = 1/16$ and $a_2/a_0 = 1/32$ that are found, for which the series (44) is annihilated by a linear ODE.

For the particular value $a_2 = a_0/16$, the linear ODE found is of *order two*, with the non logarithmic solution

$$C_{scal}^{(III)}(x) = x^{1/4} \cdot B_0\left(\frac{x}{2}\right). \quad (46)$$

For the particular value $a_2 = a_0/32$, the linear ODE is of *order one* and solves as:

$$C_{scal}^{(IV)}(x) = x^{1/4} \cdot \exp\left(\frac{x^2}{32}\right). \quad (47)$$

5.2. Correspondence with solutions of PVI sigma form equation

In the scaling limit, we have obtained that Painlevé VI sigma form equation has the following solutions:

$$\begin{aligned} C_{scal}^{(I)}(x) &= x^\alpha \cdot \exp(\pm i \frac{4\alpha - 1}{8\sqrt{\alpha}} x), & \alpha \neq 1/4, \\ C_{scal}^{(II)}(x) &= x^{1/4}, & C_{scal}^{(III)}(x) &= x^{1/4} \cdot B_0(\frac{x}{2}), \\ C_{scal}^{(IV)}(x) &= x^{1/4} \cdot \exp\left(\frac{x^2}{32}\right). \end{aligned} \quad (48)$$

For the logarithmic solution of the Painlevé VI sigma form in the scaling limit, the first terms are given in [2]. More terms are given in Appendix E.

Now, we show the solutions of the Painlevé VI sigma form corresponding to these scaling solutions, $C_{scal}^{(I)}(x), \dots, C_{scal}^{(IV)}(x)$. There is one solution to the Painlevé VI sigma form which reads

$$C(t) = t^\alpha \cdot (t-1)^\beta, \quad (49)$$

with (for $T > T_c$)

$$\alpha = -\frac{1}{8} - \frac{1}{2}\beta \pm \frac{4\beta - 1}{8\beta} \cdot \sqrt{\beta \cdot (\beta - N^2)}, \quad (50)$$

and (for $T < T_c$)

$$\alpha = \left(\frac{1}{2} - \frac{1}{8\beta}\right) \cdot (-\beta \pm \sqrt{\beta \cdot (\beta - N^2)}). \quad (51)$$

In the scaling limit, the corresponding linear differential operator is (for both regimes)

$$64\beta \cdot x^2 \cdot D_x^2 - 128\beta^2 \cdot x \cdot D_x + (4\beta - 1)^2 \cdot x^2 + 64\beta^2 \cdot (\beta + 1), \quad (52)$$

with solutions

$$c_1 \cdot x^\beta \cdot \exp\left(i \frac{4\beta - 1}{8\sqrt{\beta}} x\right) + c_2 \cdot x^\beta \cdot \exp\left(-i \frac{4\beta - 1}{8\sqrt{\beta}} x\right), \quad (53)$$

which are the solutions (43).

The same solution to the Painlevé VI sigma form can be seen as given with α being a free parameter, i.e.

$$C(t) = t^\alpha \cdot (t-1)^\beta, \quad (54)$$

with (for $T > T_c$)

$$\beta = \frac{1}{4N^2 + 16\alpha + 4} \cdot \left(N^2 - 8\alpha^2 - 2\alpha \pm (4\alpha + 1) \cdot \sqrt{4\alpha^2 - N^2}\right), \quad (55)$$

and (for $T < T_c$)

$$\beta = \frac{1}{4 \cdot (N^2 + 4\alpha)} \cdot \left(N^2 - 8\alpha^2 + 2\alpha \pm 2\alpha \sqrt{(4\alpha^2 - 1)^2 - 4N^2}\right). \quad (56)$$

In the scaling limit, the corresponding linear differential operator is (for both regimes)

$$16x^2 \cdot D_x^2 + 8x \cdot D_x + 1, \quad (57)$$

with solutions

$$c_1 \cdot x^{1/4} + c_2 \cdot x^{1/4} \cdot \ln(x), \quad (58)$$

giving the solution (45).

We have shown in [16] that any combination of the two solutions of (with D_t the derivative d/dt)

$$L_h = D_t^2 + \left(\frac{1}{t} + \frac{1}{2(t-1)}\right) \cdot D_t - \frac{1}{4} \frac{N^2}{t^2} + \frac{1}{16(t-1)^2}, \quad (59)$$

actually satisfies the Painlevé VI sigma form (35). In the scaling limit, the two solutions are

$$c_1 \cdot x^{1/4} \cdot B_0\left(\frac{x}{2}\right) + c_2 \cdot x^{1/4} \cdot K_0\left(\frac{x}{2}\right), \quad (60)$$

i.e. the scaling solution (46). Note that $x^{1/4} \cdot K_0(x/2)$ is *also a solution of* (41), and L_h annihilates $(1-t)^{1/4} \cdot f_N^{(1)}$.

6. Scaling limit of the diagonal correlation functions $C(N, N)$

Now, let us show that the scaling solution (47)

$$C_{scal}^{(IV)}(x) = x^{1/4} \cdot \exp\left(\frac{x^2}{32}\right), \quad (61)$$

corresponds (up to $x^{1/4}$) to an infinite sum of the scaling limit of the $f_N^{(j)}$, i.e. this is the scaling solution (analytical at $x = 0$) of $C(N, N)$.

We will consider $f_N^{(1)}$, $f_N^{(3)}$ and $f_N^{(5)}$, which are solutions of respectively L_2 , $L_4 \cdot L_2$, and $L_6 \cdot L_4 \cdot L_2$. These linear differential operators are given in [5], and we call L_2^{scal} , $(L_4^{scal} \cdot L_2^{scal})$ and $(L_6^{scal} \cdot L_4^{scal} \cdot L_2^{scal})$ the corresponding scaling operators.

The function $C_{scal}^{(IV)}(x)$ expands as:

$$\frac{C_{scal}^{(IV)}(x)}{x^{1/4}} = 1 + \frac{x^2}{32} + \frac{x^4}{2048} + \frac{x^6}{2^{16}3} + \frac{x^8}{2^{23}3} + \frac{x^{10}}{2^{28}15} + \frac{x^{12}}{2^{34}45} + \dots \quad (62)$$

The identification will be done on the formal solutions of the scaling linear differential operators.

With the first terms of the solution of L_2^{scal}

$$S_2 = 1 + \frac{x^2}{16} + \frac{x^4}{1024} + \frac{x^6}{147456} + \dots, \quad (63)$$

there is only the constant term which matches.

The analytical solution, at $x = 0$, of $L_4^{scal} \cdot L_2^{scal}$ reads:

$$\begin{aligned} S_{42} = & 1 + a_2 \cdot x^2 + \frac{a_2}{64} \cdot x^4 + \frac{1}{2^{14} \cdot 3^2} \cdot x^6 \\ & + \left(\frac{1}{2^{20} \cdot 3^2} - \frac{a_2}{2^{18} \cdot 3} \right) \cdot x^8 + \dots \end{aligned} \quad (64)$$

With the well suited combination ($a_2 = 1/32$), S_{42} becomes:

$$S_{42} = 1 + \frac{x^2}{32} + \frac{x^4}{2048} + \frac{x^6}{2^{14}3^2} + \dots \quad (65)$$

We see that, up to x^4 , the coefficients are reproduced, i.e. up to x^4 , the solution $C_{scal}^{(IV)}(x)$ is reproduced by the scaling limit of:

$$(1-t)^{1/4} \cdot \left(f_N^{(1)} + f_N^{(3)}\right). \quad (66)$$

Note that the next coefficients of $C_{scal}^{(IV)}(x)/x^{1/4}$, and S_{42} (i.e. at x^6), are in the ratio 4/3.

Next, we consider the scaling of

$$(1-t)^{1/4} \cdot (f_N^{(1)} + f_N^{(3)} + f_N^{(5)}). \quad (67)$$

This amounts to considering the solution of $L_6^{scal} \cdot L_4^{scal} \cdot L_2^{scal}$

$$\begin{aligned} S_{642} = & 1 + a_2 \cdot x^2 + \frac{a_2}{64} \cdot x^4 + a_6 \cdot x^6 + \left(\frac{a_6}{64} - \frac{a_2}{2^{18} \cdot 3} \right) \cdot x^8 \\ & + \left(\frac{13a_6}{2^{15} \cdot 25} - \frac{a_2}{2^{20} \cdot 75} \right) \cdot x^{10} \\ & + \left(\frac{49a_6}{2^{18} \cdot 3^2 \cdot 5^2} - \frac{11a_2}{2^{26} \cdot 3^4 \cdot 5^2} - \frac{1}{2^{30} \cdot 3^4 \cdot 5^2} \right) \cdot x^{12} + \dots, \end{aligned} \quad (68)$$

and obtaining the well suited combination ($a_2 = 1/32$, $1/a_6 = 2^{16} \cdot 3$)

$$S_{642} = 1 + \frac{x^2}{32} + \frac{x^4}{2048} + \frac{x^6}{2^{16} \cdot 3} + \frac{x^8}{2^{23} \cdot 3} + \frac{x^{10}}{2^{28} \cdot 15} + \frac{43x^{12}}{2^{34} \cdot 3^4 \cdot 5^2} + \dots \quad (69)$$

which reproduces $C_{scal}^{(IV)}(x)/x^{1/4}$ up to x^{10} , the ratio of the next coefficients 43/45 being almost the unity.

With the first three form factors, we may infer that, for each $f_N^{(2n+1)}$ form factor added to $C(N, N)$, the coefficients of the scaling function are reproduced up to $x^{n(n+3)}$.

Indeed, and as a last check, we consider the next form factor $f_N^{(7)}$ whose scaling limit is given by $L_8^{scal} \cdot L_6^{scal} \cdot L_4^{scal} \cdot L_2^{scal}$, and its analytical solution (at $x = 0$) which reads:

$$\begin{aligned} S_{8642} = & 1 + a_2 \cdot x^2 + \frac{a_2}{64} x^4 + a_6 \cdot x^6 + \left(\frac{a_6}{64} - \frac{a_2}{2^{18} \cdot 3} \right) \cdot x^8 \\ & + \left(\frac{13a_6}{2^{15} \cdot 25} - \frac{a_2}{2^{20} \cdot 75} \right) \cdot x^{10} + a_{12} \cdot x^{12} \\ & + \left(\frac{a_{12}}{64} - \frac{33a_6}{2^{24} \cdot 7^2 \cdot 5} + \frac{a_2}{2^{29} \cdot 3^2 \cdot 5 \cdot 7^2} \right) \cdot x^{14} + \dots \end{aligned} \quad (70)$$

With the well suited combination $a_2 = 1/32$, $1/a_6 = 2^{16} \cdot 3$, $1/a_{12} = 2^{34} \cdot 45$, S_{8642} reproduces $C_{scal}^{(IV)}(x)/x^{1/4}$ up to x^{18} , and the next coefficients are in the ratio 1571/1575.

Note that we have the same results when we consider the scaling limits of $f_N^{(n)}$ with n even. For this, let us show the analytical solution, at $x = 0$, for $L_5^{scal} \cdot L_3^{scal} \cdot L_1^{scal}$, which reads

$$\begin{aligned} S_{531} = & 1 + a_2 \cdot x^2 + \frac{a_2}{64} \cdot x^4 + a_6 \cdot x^6 + \left(\frac{a_6}{64} - \frac{a_2}{2^{18} \cdot 3} \right) \cdot x^8 \\ & + \left(\frac{13a_6}{2^{15} \cdot 25} - \frac{a_2}{2^{20} \cdot 75} \right) \cdot x^{10} + \left(\frac{37a_6}{2^{18} \cdot 3^2 \cdot 5^2} - \frac{a_2}{2^{23} \cdot 3^4 \cdot 5^2} \right) \cdot x^{12} + \dots, \end{aligned} \quad (71)$$

where we remark that this solution identifies with S_{642} , up to x^{10} , i.e. for the *same* well suited combination it reproduces $C_{scal}^{(IV)}(x)/x^{1/4}$ up to x^{10} . In other words the scaling limit of $f_N^{(1)} + f_N^{(3)} + \dots + f_N^{(2n+1)}$ identifies with the scaling limit of $1 + f_N^{(2)} + f_N^{(4)} + \dots + f_N^{(2n)}$, up to $x^{n(n+3)}$. As far as the analytical solution at $x = 0$ of the scaling function is concerned, the scaling function is the *same for both regimes* (high and low temperatures).

The scaling limit of $C(N, N)$ is therefore:

$$\lim_{t \rightarrow 1, N \rightarrow \infty} (1-t)^{1/4} \cdot \sum_{n: \text{odd, even}} f_N^{(n)}(t) = x^{1/4} \cdot \exp\left(\frac{x^2}{32}\right). \quad (72)$$

7. Scaling limit of the next-to-diagonal correlation functions $C(N, N+1)$

The non-diagonal correlation functions $C(N, M)$ are given in terms of determinants (see [7]). It has been shown in [17] that the next-to-diagonal correlation functions $C(N, N+1)$ have the form of a bordered Toeplitz determinant. An iteration scheme of the diagonal and the next-to-diagonal correlation functions is given by Witte [18].

Unlike the diagonal correlation functions $C(N, N)$ which are annihilated by Painlevé VI equation, there is no known (non-linear) differential equation for $C(N, N+1)$ on which the simple scaling limit $t \rightarrow 1, N \rightarrow \infty$ can be performed. However, these next-to-diagonal correlation functions can be written as sum of the form factors [19], $C^{(n)}(N, N+1)$. In Appendix F, we show that these next-to-diagonal form factors are annihilated by linear ODEs that can be obtained for *generic* N . We give in Appendix F the first three linear differential operators and their corresponding linear differential operators in the scaling limit.

It appears that these linear differential operators, in the scaling limit, identify with the operators for the diagonal $f_N^{(n)}$ in the scaling limit. Therefore, we will expect the occurrence of the same expression $x^{1/4} \cdot \exp(x^2/32)$ as the scaling limit of $C(N, N+1)$.

Consider the first term $C^{(1)}(N, N+1)$ whose scaling limit is given by the direct sum $L_2^{scal} \oplus L_1^{scal}$, which has the analytic solution at $x = 0$:

$$a_0 + a_2 \cdot x^2 + \frac{1}{64} a_2 \cdot x^4 + \frac{1}{9216} a_2 \cdot x^6 + \frac{1}{2359296} a_2 \cdot x^8 + \dots \quad (73)$$

For $a_0 = 1$ and $a_2 = 1/32$, there is matching with $\exp(x^2/32)$ up to x^4 .

The two terms $C^{(1)}(N, N+1) + C^{(3)}(N, N+1)$ are annihilated by the operator \mathcal{V}_{10} which solves $C^{(2)}(N, N+1)$ as well. But we have shown in Appendix F that in the scaling limit, the operator \mathcal{V}_{10} has the *direct sum* decomposition (F.14). This allows us to pick only the operators $L_1^{scal} \oplus L_4^{scal} \cdot L_2^{scal}$ corresponding to the scaling limit of $C^{(1)}(N, N+1) + C^{(3)}(N, N+1)$. The analytic solution at $x = 0$ expands as

$$\begin{aligned} a_0 + a_2 \cdot x^2 + \frac{1}{64} a_2 \cdot x^4 + a_6 \cdot x^6 + \left(\frac{a_6}{64} - \frac{a_2}{786432}\right) \cdot x^8 \\ + \left(\frac{13a_6}{102400} - \frac{a_2}{78643200}\right) \cdot x^{10} + \dots \end{aligned} \quad (74)$$

and for $a_0 = 1, a_2 = 1/32, a_6 = 1/196608$, matches with $\exp(x^2/32)$ up to x^{10} .

Let us consider the whole solutions of the scaling limit of \mathcal{V}_{10} , which means that we are matching $\exp(x^2/32)$ to the scaling limit of $C^{(1)}(N, N+1) + C^{(2)}(N, N+1) + C^{(3)}(N, N+1)$, i.e. mixing both regimes. The analytic solution at $x = 0$ of (F.14) depends on *four free coefficients*, which when fixed to $a_0 = 1, a_2 = 1/32, a_6 = 1/196608, a_{12} = 1/773094113280$, actually matches $\exp(x^2/32)$ up to x^{18} .

Therefore, we have shown that

$$\lim_{s \rightarrow 1, N \rightarrow \infty} s^{-1} \cdot (1-s^4)^{1/4} \cdot \sum_{n: \text{odd, even}} C^{(n)}(N, N+1) = x^{1/4} \cdot \exp\left(\frac{x^2}{32}\right). \quad (75)$$

8. Differential Galois groups of the operators in the scaling limit

The equivalence of two properties, namely the *homomorphism of the operator with its adjoint*, and either the occurrence of a *rational solution* for the *symmetric* (or *exterior*) square of that operator, or the drop of order of these squares[†], have been seen for many linear differential operators [20].

The linear differential operators with these properties are such that their differential Galois groups are included in the symplectic, or orthogonal, differential groups.

The solutions of the operators L_n , when N is fixed to an integer, write as *polynomials* in the complete elliptic integrals K and E . The operators are equivalent to some symmetric power of L_E , the linear differential operator for the complete elliptic integral E . The homomorphisms of the L_n with their corresponding adjoint is, therefore, a straightforward consequence of the homomorphism of L_E with its adjoint.

Furthermore, we forwarded, in a recent paper [21], a “canonical decomposition” for those operators whose *differential Galois groups* are included in *symplectic* or *orthogonal* groups. These linear differential operators are homomorph to their adjoints, and a “canonical decomposition” of these linear differential operators can be written in terms of a “tower of intertwiners” [21].

The issue, we address in the sequel, is whether these properties hold for the operators L_n with a *generic parameter* N , and whether this is preserved in the scaling limit.

We find that the linear differential operators L_n (given up to $n = 10$ in [5]) are homomorphic to their respective adjoints for *generic values* of N . Their differential Galois groups are in *symplectic groups* for n even, and in *orthogonal groups* for n odd. Their exterior (for n even) and symmetric (for n odd) squares do annihilate a *rational function*. For instance, the rational solution of the *symmetric square* of L_3 reads

$$\text{sol}_R(\text{sym}^2(L_3)) = \frac{N^2 \cdot t^2 - (2N^2 - 1) \cdot t + N^2}{t^2 \cdot (1 - t)^2}, \quad (76)$$

and the rational solution of the *exterior square* of L_4 is:

$$\text{sol}_R(\text{ext}^2(L_4)) = \frac{(N^2 - 1) \cdot t^2 - 2N^2 \cdot t + (N^2 - 1)}{t^5 \cdot (1 - t)^3}. \quad (77)$$

The operator L_3^{scal} is the scaling limit of L_3 , and is (non-trivially) *homomorphic to its adjoint*. The rational solution of its *symmetric square* is

$$\text{sol}_R(\text{sym}^2(L_3^{\text{scal}})) = \frac{1 + x^2}{x^2}, \quad (78)$$

which is the rational function (76) in the scaling limit.

Similarly, the operator L_4^{scal} (the scaling limit of L_4) is homomorphic to its adjoint, and the rational solution of its *exterior square* reads

$$\text{sol}_R(\text{ext}^2(L_4^{\text{scal}})) = \frac{x^2 - 2}{x^3}, \quad (79)$$

which, in the scaling limit, is the rational function (77).

[†] The order of the symmetric (or exterior) of these operators is less than the order generically expected for these squares. In terms of differential systems this corresponds, however, to rational solutions.

The orthogonal (resp. symplectic) differential Galois groups admit an invariant quadratic (resp. alternating) form. Here also, for instance for L_3 , one has the following quadratic form, depending on N , $Q(X_0, X_1, X_2) = \text{const.}$ where

$$\begin{aligned} Q(X_0, X_1, X_2) = & \left(t^2 \cdot (5 - 10t + 4t^2) - (1-t)^4 \cdot N^2 \right) \cdot X_0^2 \\ & + \left(t^2 (1-t)^2 (4 - 17t + 16t^2) - t^2 \cdot (1-t)^4 \cdot N^2 \right) \cdot X_1^2 \\ & + t^4 \cdot (1-t)^4 \cdot X_2^4 - \left(2t \cdot (1-t)^4 N^2 + t^2 \cdot (1-t)(9 - 27t + 16t^2) \right) \cdot X_0 X_1 \\ & - 4t^3 \cdot (1-t)^3 \cdot X_2 X_0 + 4t^3 \cdot (1-2t)(1-t)^3 \cdot X_2 X_1, \end{aligned} \quad (80)$$

which, in the scaling limit, becomes the quadratic form

$$\begin{aligned} Q^{scal}(X_0, X_1, X_2) = & X_0^2 - x^2 \cdot (3 - x^2) \cdot X_1^2 - x^4 \cdot X_2^2 \\ & - 4x^3 \cdot X_1 X_2 + 2x \cdot X_0 X_1, \end{aligned} \quad (81)$$

for L_3^{scal} . In (80), and (81), X_0 denotes any solution of the considered linear differential operator, X_1 and X_2 being the first and second derivative of X_0 .

The operators L_n^{scal} “inheritate” the differential Galois groups of the operators L_n . For n even (resp. n odd), the differential Galois group of L_n^{scal} is *included* in $Sp(n, \mathbb{C})$ (resp. $SO(n, \mathbb{C})$). Recall that the solutions of the operators L_n (resp. L_n^{scal}) write as polynomials in the complete elliptic integrals (resp. modified Bessel functions), which means that the linear differential operators L_n (resp. L_n^{scal}) are homomorphic[‡] to the symmetric $(n-1)$ -th power of L_2 (resp. L_2^{scal}). Thus, the differential Galois group of L_n and L_n^{scal} is, in fact, the differential Galois group of L_2 (or L_2^{scal}), namely[†] $SL(2, \mathbb{C})$.

We have shown in [21] that the homomorphism of the operator with its adjoint implies a “canonical decomposition” in terms of *self-adjoint operators*. This decomposition is obtained by a sequence of Euclidean right divisions (see [21] and section 9 in [15]). The operator L_3 has the canonical decomposition (for generic N)

$$L_3 = (U_1^{(3)} \cdot U_2^{(3)} \cdot U_3^{(3)} + U_1^{(3)} + U_3^{(3)}) \cdot r_1^{(3)}(x), \quad (82)$$

where $r_1^{(3)}(x)$ is a rational function, and where $U_1^{(3)}$, $U_2^{(3)}$ and $U_3^{(3)}$ are *order-one self-adjoint* operators. In the scaling limit, one obtains for L_3^{scal} :

$$L_3^{scal} = W_1^{(3)} \cdot W_2^{(3)} \cdot W_3^{(3)} + W_1^{(3)} + W_3^{(3)} \cdot r_2^{(3)}(x). \quad (83)$$

Here also, $r_2^{(3)}(x)$ is a rational function, and $W_1^{(3)}$, $W_2^{(3)}$ and $W_3^{(3)}$ are *order-one self-adjoint* operators.

Similarly, the operator L_4 has the following canonical decomposition (for generic N)

$$L_4 = (U_1^{(4)} \cdot U_2^{(4)} + 1) \cdot r_1^{(4)}(x), \quad (84)$$

where $r_1^{(4)}(x)$ is a rational function, and where $U_1^{(4)}$ and $U_2^{(4)}$ are *order-two self-adjoint* operators. In the scaling limit, one obtains for L_4^{scal}

$$L_4^{scal} = (W_1^{(4)} \cdot W_2^{(4)} + 1) \cdot r_2^{(4)}(x), \quad (85)$$

[‡] L_n is homomorphic to the symmetric $(n-1)$ -th power of L_2 , with N generic (not necessarily an integer).

[†] $SL(2, \mathbb{C})$ is isomorphic to $Sp(2, \mathbb{C})$, to $Spin(3, \mathbb{C})$, and isomorphic, up to a 2-to-1 homomorphism, to $SO(3, \mathbb{C}) \simeq PSL(2, \mathbb{C})$.

where $r_2^{(4)}(x)$ is a rational function, and where $W_1^{(4)}$ and $W_2^{(4)}$ are *order-two self-adjoint* operators.

The “canonical” decomposition occurring for the operators L_n , is *preserved in the scaling limit*. In particular the *self-adjoint* operators of these “canonical” decompositions [21] are *all of order one* for the L_n and L_n^{scal} with n odd and are *all of order two* for the L_n and L_n^{scal} with n even. The rational solutions of the symmetric, or exterior, squares of the L_n^{scal} are given in Appendix G.

9. Conclusion

To obtain the expression $x^{1/4} \cdot \exp(x^2/32)$ as the scaling limit of the correlation functions $C(N, N)$, we have made a “matching”, in the scaling limit, of both hand-sides of:

$$C(N, N) = (1 - t)^{1/4} \cdot \sum_j f_N^{(j)}. \quad (86)$$

The left-hand-side is taken as a particular solution that pops out from the *sigma form* of Painlevé VI in the scaling limit. The right-hand-side is a particular combination of the sum of the (non logarithmic) formal solutions of the operators (annihilating $f_N^{(j)}$) at scaling.

For the next-to-diagonal correlation functions $C(N, N + 1)$, there is no (non-linear) differential equation one can use, but we have obtained that the next-to-diagonal form factors $C^{(j)}(N, N + 1)$ have, in the scaling limit, the *same* linear differential operators L_n^{scal} . One may conjecture that we will obtain the same linear differential operators at scaling for the j -contributions $C^{(j)}(N, N + p)$, $C^{(j)}(N, p \cdot N)$ with $p > 1$ or $C^{(j)}(N, M)$.

Each time the discrete parameter N of the lattice appears explicitly in a differential equation, the scaling limit can easily be performed. The correlation functions $C(N, N)$ is a solution of the sigma form of Painlevé VI (see (35)) which, itself, is a specialisation of a more general nonlinear differential equation [22, 23], also called sigma form of Painlevé VI, which depends on *four* parameters†. The scaling limit performed on (35) with (36) has given the nonlinear equation (41) that identifies with (10) which concerns the scaling limit of the correlation functions $C(M, N)$. If one assumes that, similarly to $C(N, N)$, the $C(M, N)$ also verify a non-linear differential equation, generalizing (35), one possible scenario could be that a two-parameter nonlinear equation for $C(M, N)$ emerges as a subcase of the four-parameter sigma form of Painlevé VI. Finding this two-parameter nonlinear equation for $C(M, N)$ essentially requires to generalize the definitions of σ , namely (36), and to find the constraints on the four parameters.

The square Ising model has shown an extremely rich structure illustrated by a large set of exact results corresponding to highly selected linear differential equations of the n -particle contribution to the magnetic susceptibility $\chi^{(n)}$, correlation functions $C(N, M)$, form factors $C^{(j)}(N, M)$, etc. For the linear ODE which have only the three§ regular singularities $t = 0$, $t = 1$ and $t = \infty$, the scaling limit leads to a

† The general Painlevé VI sigma form (eq.(1) in [16]), deals with the function $\zeta(t)$ and depends on *four* parameters v_1, \dots, v_4 . Equation (35) for the $C(N, N)$ is the subcase, $\sigma(t) = \zeta(t) + N^2 \cdot t/4 - 1/8$, $v_1 = v_4 = N/2$, $v_2 = (1 - N)/2$ and $v_3 = (1 + N)/2$.

§ This is at contrast with, for example, the case of the magnetic susceptibility of the Ising model which is an *infinite sum* of contributions with large set of regular singularities that eventually densify the

confluence [25, 26] of the singularities, ending in the regular $x = 0$ and the *irregular* $x = \infty$ points.

All the remarkable structures discovered in previous papers, on the square Ising model (elliptic functions, modular forms, Calabi-Yau equations, “special” differential Galois groups, globally bounded series, diagonals of rational functions, ...) emerge in a framework related to the (Yang-Baxter) integrability concept occurring *on a lattice*. In the scaling limit, with the emergence of irregular singularities from the confluence of regular ones, many of these structures actually disappear, or are less crystal clear. For instance, the property of global nilpotence, occurring in all our linear ODEs, disappear in the scaling limit, but some structures still show up for the p -curvature (see section 10 in [27]). In contrast, we have seen that the differential Galois group structures are more robust, being preserved by the scaling limit.

What happens in the scaling limit to *all* the remarkable holomic or non-holonomic structures we have discovered in the last decade, on the square Ising model?

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Appendix A. Recall: $C(N, N)$ and $f_N^{(n)}$ as polynomials in K and E

The correlation functions $C(N, N)$ are the analytical (at 0) solutions of linear ODE of order $N + 1$. For N fixed to an integer, the correlation functions $C(N, N)$ writes as *polynomials* in the complete elliptic integrals of first and second kind K and E of homogeneous degree N . With

$$K = {}_2F_1([1/2, 1/2], [1], t), \quad E = {}_2F_1([1/2, -1/2], [1], t), \quad (\text{A.1})$$

the form of $C(N, N)$ reads

$$C(N, N) = \sum_{i=0}^N Q(N, i, t) \cdot K^{N-i} \cdot E^i, \quad (\text{A.2})$$

where $Q(N, i, t)$ is a rational function. For instance $C(2, 2)$ in the $T > T_c$ regime, writes:

$$3 \cdot t \cdot C(2, 2) = 3 \cdot (t - 1)^2 \cdot K^2 + 8 \cdot (t - 1) \cdot K \cdot E - (t - 5) \cdot E^2. \quad (\text{A.3})$$

The form factors $f_N^{(n)}$ are the analytical (at 0) solutions of linear differential operators with N as a parameter. With n and N fixed to integers, $f_N^{(n)}$ writes as a *sum of polynomials* in K and E . The form of $f_N^{(2n+1)}$ reads

$$f_N^{(2n+1)} = \sum_{j=0}^n \sum_{i=0}^{2j+1} P(N, n, j, i, t) \cdot K^{2j+1-i} \cdot E^i, \quad (\text{A.4})$$

with $P(N, n, j, i, t)$ a rational function. In the expression of $f_N^{(2n+1)}$, the homogeneous degrees of K and E occur as $1, 3, \dots, 2n+1$. Recall [5] that the linear differential operators annihilating the $f_N^{(2n+1)}$, have a *direct sum* structure when the

unit circle $|s| = 1$ yielding a *natural boundary*. For the scaling function of the magnetic susceptibility χ , see [24] and references therein.

parameter N is fixed to an integer. The first two $f_N^{(2n+1)}$ contributing to the example of $C(2, 2)$ are:

$$3t \cdot f_2^{(1)} = t \cdot (t+2) \cdot K - 2t \cdot (t+1) \cdot E, \quad (\text{A.5})$$

$$\begin{aligned} 18t \cdot f_2^{(3)} = & -3 \cdot (t^2 - 2) \cdot K^3 + 3 \cdot (2t^2 - 11t + 2) \cdot K^2 \cdot E \\ & + 36 \cdot (t-1) \cdot K \cdot E^2 + 24E^3 + 7 \cdot (t+2) \cdot K - 14 \cdot (t+1) \cdot E. \end{aligned} \quad (\text{A.6})$$

The expression (16) reproduced here for $N = 2$

$$C(2, 2) = (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} f_2^{(2n+1)}, \quad (\text{A.7})$$

shows that an *infinite sum* of polynomials in K and E will give birth to the overall factor $(1-t)^{-1/4}$ absent in (A.3). This situation has been encountered in the magnetic susceptibility of Ising model at scaling (see section 7 in [28]). See also section 5.1 in [29], where a sum of terms, each term being a polynomial expression of the complete elliptic integrals, reduces to an algebraic expression.

Appendix B. Recall of the expressions of L_n^{scal} , $n = 1, 2, \dots, 6$

The form factors $f_N^{(1)}$ and $f_N^{(3)}$ are annihilated by the order-six operator $L_4 \cdot L_2$, which, in the scaling limit, writes $L_4^{scal} \cdot L_2^{scal}$, where:

$$\begin{aligned} L_2^{scal} &= 4x \cdot D_x^2 + 4D_x - x, \\ L_4^{scal} &= 16x^3 \cdot D_x^4 + 160x^2 \cdot D_x^3 - 8x \cdot (5x^2 - 46) \cdot D_x^2 \\ &\quad - 72 \cdot (x^2 - 2) \cdot D_x + 9x^3. \end{aligned} \quad (\text{B.1})$$

The form factor $f_N^{(5)}$ is annihilated by the order-twelve linear differential operator $L_6 \cdot L_4 \cdot L_2$, which, in the scaling limit, writes $L_6^{scal} \cdot L_4^{scal} \cdot L_2^{scal}$, where L_6^{scal} reads:

$$\begin{aligned} L_6^{scal} &= 64x^5 \cdot D_x^6 + 2240x^4 \cdot D_x^5 - 112x^3 \cdot (5x^2 - 236) \cdot D_x^4 \\ &\quad - 32x^2 \cdot (259x^2 - 3916) \cdot D_x^3 + 4x \cdot (259x^4 - 7668x^2 + 54128) \cdot D_x^2 \\ &\quad + 100 \cdot (784 - 236x^2 + 27x^4) \cdot D_x - 225x^5. \end{aligned} \quad (\text{B.2})$$

The form factor $f_N^{(2)}$ is annihilated by the order-four operator $L_3 \cdot L_1$, which, in the scaling limit, writes $L_3^{scal} \cdot L_1^{scal}$, where:

$$\begin{aligned} L_1^{scal} &= D_x, \\ L_3^{scal} &= 2x^3 \cdot D_x^3 + 8x^2 \cdot D_x^2 - 2(x-1)(x+1) \cdot x \cdot D_x - 2. \end{aligned} \quad (\text{B.3})$$

Note that $L_3^{scal} \cdot L_1^{scal}$ has a direct sum decomposition $L_3^{scal} \cdot L_1^{scal} = L_1^{scal} \oplus \tilde{L}_3^{scal}$, with

$$\tilde{L}_3^{scal} = x^2 \cdot D_x^3 + 3x \cdot D_x^2 + (1-x^2) \cdot D_x + x. \quad (\text{B.4})$$

The form factor $f_N^{(4)}$ is annihilated by the order-nine operator $L_5 \cdot L_3 \cdot L_1$, which, in the scaling limit, writes $L_5^{scal} \cdot L_3^{scal} \cdot L_1^{scal}$, where L_5^{scal} reads:

$$\begin{aligned} L_5^{scal} &= 2x^5 \cdot D_x^5 + 40x^4 \cdot D_x^4 - 2x^3 \cdot (5x^2 - 113) \cdot D_x^3 - 2x^2 \cdot (32x^2 - 161) \cdot D_x^2 \\ &\quad + 2x \cdot (4x^4 - 97 - 24x^2) \cdot D_x + 32x^2 - 256. \end{aligned} \quad (\text{B.5})$$

Appendix C. $I_2(x)$ again

The choice of the basis of the formal solutions is arbitrary. Instead of the basis $(S_1^{(2)}, S_2^{(2)}, S_3^{(2)}, S_4^{(2)})$, one may take

$$\begin{aligned}\tilde{S}_1^{(2)} &= S_1^{(2)} + S_2^{(2)} - \frac{15}{2} S_3^{(2)} + \frac{17}{2} S_4^{(2)}, \\ \tilde{S}_2^{(2)} &= S_2^{(2)} + \frac{39}{16} S_3^{(2)} - \frac{31}{16} S_4^{(2)}, \quad \tilde{S}_3^{(2)} = S_3^{(2)}, \quad \tilde{S}_4^{(2)} = S_4^{(2)},\end{aligned}\tag{C.1}$$

where the series begin, now, as $const. + \dots$. The combination coefficients $\tilde{c}_j^{(2)}$ will appear as

$$\tilde{c}_1^{(2)} = 0.1013211, \quad \tilde{c}_2^{(2)} = -0.06263, \quad \tilde{c}_3^{(2)} = 0.61863, \quad \tilde{c}_4^{(2)} = -0.53296,$$

and in exact forms as:

$$\begin{aligned}\tilde{c}_1^{(2)} &= \frac{1}{\pi^2}, & \tilde{c}_2^{(2)} &= \frac{1}{\pi^2} \cdot (1 - 4 \ln(2) + 2\gamma), \\ \tilde{c}_3^{(2)} &= \frac{1}{4\pi^2} \cdot (1 - 4 \ln(2) + 2\gamma)^2 + \frac{1}{8\pi^2} \cdot (23 + 62 \ln(2) - 31\gamma), \\ \tilde{c}_4^{(2)} &= -\frac{1}{8\pi^2} \cdot (17 + 62 \ln(2) - 31\gamma).\end{aligned}\tag{C.2}$$

Appendix D. The $C(N, N)$ correlation functions

The correlation functions $C(N, N)$ are annihilated by a linear ODE of order $N + 1$. The form of the linear differential operators is

$$L_{N+1} = P_{N+1} \cdot D_x^{N+1} + P_N \cdot D_x^N + \dots + P_0, \tag{D.1}$$

where, for generic N , the first polynomials P_{N-k} (for $N > k$) read:

$$\begin{aligned}P_{N+1} &= x^{N+1} \cdot (x-1)^N, \\ P_N &= -\frac{1}{6} \cdot x^N (x-1)^{N-1} \cdot N \cdot (N+1) \cdot \left((N-4) \cdot x + (N+2) \right), \\ P_{N-1} &= \frac{1}{260} \cdot x^{N-1} \cdot (x-1)^{N-2} \cdot N(N+1) \left((N-1)(N-2) \cdot (5N^2 - 26N + 18) \cdot x^2 \right. \\ &\quad \left. + (N+2) \cdot (10N^3 - 54N^2 + 62N - 3) \cdot x \right. \\ &\quad \left. + (N+2)(5N^3 + 9N^2 - 32N + 3) \right), \\ P_{N-2} &= \frac{1}{45360} \cdot x^{N-2} \cdot (x-1)^{N-3} \cdot N(N+1)(N-1) \times \\ &\quad \left((N-2) \cdot (35N^5 - 371N^4 + 1564N^3 - 3676N^2 + 4320N - 2448) \cdot x^3 \right. \\ &\quad \left. + 3(N-3) \cdot (N+2)(35N^4 - 280N^3 + 772N^2 - 929N + 135) \cdot x^2 \right. \\ &\quad \left. + 3 \cdot (N+2) \cdot (35N^5 - 175N^4 - 194N^3 + 2110N^2 - 2748N + 603) \cdot x \right. \\ &\quad \left. + (N+2) \cdot (35N^5 + 119N^4 - 578N^3 - 1175N^2 + 2682N - 954) \right).\end{aligned}\tag{D.2}$$

Appendix E. Non analytical scaling of $C(N, N)$

Seeking a logarithmic solution of (41), one obtains two solutions that depend on the parameter e_1

$$C_{scal}(x) = \text{const.} \cdot \sum_{k=0}^{\infty} (\pm 1)^k \cdot S_k(\pm x) \cdot \left(\pm \frac{1}{4} \ln(x) + e_1 \right)^k, \quad (\text{E.1})$$

The matching with the first terms given in [2], fixes the parameter $e_1 = \ln(2) - \gamma/4$.

The first $S_k(x)$ read:

$$\begin{aligned} S_0(x) &= 1 + \frac{1}{64} \cdot x^2 + \frac{1}{2^{15}} \cdot x^4 - \frac{1}{2^{17}} \cdot x^5 - \frac{5}{2^{21} \cdot 3} \cdot x^6 - \frac{1}{2^{23}} \cdot x^7 - \frac{469}{2^{34} \cdot 3} \cdot x^8 + \dots, \\ S_1(x) &= x + \frac{1}{64} \cdot x^3 + \frac{1}{2^{10}} \cdot x^4 + \frac{5}{2^{15}} \cdot x^5 + \frac{1}{2^{16}} \cdot x^6 + \frac{7}{2^{21} \cdot 3} \cdot x^7 + \frac{35}{2^{28}} \cdot x^8 + \dots, \\ S_2(x) &= -\frac{1}{2^8} x^4 - \frac{1}{2^{14}} \cdot x^6 - \frac{17}{2^{25}} x^8 + \frac{5}{2^{29}} x^9 - \frac{19}{2^{31} \cdot 3} \cdot x^{10} + \dots, \\ S_3(x) &= -\frac{1}{2^{26}} \cdot x^9 - \frac{1}{2^{32}} \cdot x^{11} - \frac{37}{2^{41} \cdot 3^2} \cdot x^{13} - \frac{13}{2^{47} \cdot 3^2} \cdot x^{15} - \frac{13}{2^{54} \cdot 3^3} \cdot x^{16} + \dots, \\ S_4(x) &= \frac{1}{2^{52} \cdot 3^2} \cdot x^{16} + \frac{1}{2^{58} \cdot 3^2} x^{18} + \frac{65}{2^{71} \cdot 3^2} \cdot x^{20} + \frac{67}{2^{77} \cdot 3^3} \cdot x^{22} + \dots, \\ S_5(x) &= \frac{1}{2^{90} \cdot 3^4} \cdot x^{25} + \frac{1}{2^{96} \cdot 3^4} \cdot x^{27} + \frac{101}{2^{105} \cdot 3^4 \cdot 5^2} \cdot x^{29} + \frac{103}{2^{111} \cdot 3^5 \cdot 5^2} \cdot x^{31} + \dots \end{aligned} \quad (\text{E.2})$$

The series $S_k(x)$ begin as $S_k(x) = A_k \cdot x^{k^2} + \dots$. At the order $k^2 + 2k$, both the even and the odd orders occur. In between x^{k^2} and x^{k^2+2k} , only the coefficients of x^{k^2+2p} occur (exception of S_0). This scheme yields that $S_k(x)$ writes as (with $k \geq 1$):

$$S_k(x) = A_k \cdot x^{k^2} \cdot \left(1 + \sum_{p=1}^k a_{2p}^{(k)} \cdot x^{2p} + \sum_{p=2k+1}^{\infty} b_p^{(k)} \cdot x^p \right) \quad (\text{E.3})$$

From the first small series of $S_k(x)$ that we have produced, we infer the following coefficients:

$$\begin{aligned} \frac{1}{A_k} &= (-1)^{k(k+3)/2} \cdot 2^{4k(k-1)} \cdot \prod_{j=1}^k \Gamma(j)^2, \\ a_2^{(k)} &= \frac{1}{64}, \quad a_4^{(k)} = \frac{1 + 4k^2}{2^{15} \cdot k^2}, \quad k > 1, \\ a_6^{(k)} &= \frac{3 + 4k^2}{2^{21} \cdot 3 \cdot k^2}, \quad k \geq 1, \quad a_8^{(k)} = \frac{51 - 58k^2 - 16k^4 + 32k^6}{2^{32} \cdot 3 \cdot k^2 (k^2 - 1)^2}, \quad k \geq 2. \end{aligned} \quad (\text{E.4})$$

Appendix F. The next-to-diagonal $C^{(j)}(N, N+1)$ Ising form factors

The form factors $C^{(j)}(N, M)$ for the anisotropic lattice, are given, in [19]. For the isotropic case the result is

$$\begin{aligned} C^{(j)}(M, N) &= \frac{1}{j!} \int_{-\pi}^{\pi} \frac{d\phi_1}{2\pi} \dots \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} \left(\prod_{n=1}^j \frac{1}{\sinh \gamma_n} \right) \\ &\times \left(\prod_{1 \leq i \leq k \leq j} h_{ik} \right)^2 \left(\prod_{n=1}^j x_n \right)^M \cos \left(N \sum_{n=1}^j \phi_n \right), \end{aligned} \quad (\text{F.1})$$

with :

$$\begin{aligned} x_n &= \frac{1}{2w} - \cos \phi_n - \left(\left(\frac{1}{2w} - \cos \phi_n \right)^2 - 1 \right)^{1/2}, \\ \sinh \gamma_n &= \left(\left(\frac{1}{2w} - \cos \phi_n \right)^2 - 1 \right)^{1/2}, \quad h_{ik} = \frac{2 (x_i x_k)^{1/2} \sin((\phi_i - \phi_k)/2)}{1 - x_i x_k}, \end{aligned} \quad (\text{F.2})$$

with $w = s/2/(1 + s^2)$, and where s denotes $\sinh(2K)$.

Appendix F.1. The linear differential equations of $C^{(j)}(N, N+1)$, $j = 1, 2, 3$

We give the linear differential equations that annihilate the first next-to-diagonal $C^{(j)}(N, N+1)$ form factors ($j = 1, 2, 3$).

The first terms of $C^{(1)}(N, N+1)$ read (with $x = w^2$)

$$\begin{aligned} C^{(1)}(N, N+1) &= \frac{2\Gamma(2+2N)}{\Gamma(1+N)\Gamma(2+N)} \cdot x^{N+1} \cdot \left(1 + \frac{2(3+2N)^2}{2+N} \cdot x \right. \\ &\quad \left. + \frac{4(3+2N)(5+2N)^2}{3+N} \cdot x^2 + \frac{8(3+2N)(5+2N)^2(7+2N)^2}{3(2+N)(4+N)} \cdot x^3 \right). \end{aligned} \quad (\text{F.3})$$

These series are annihilated by an order-three ODE whose corresponding linear differential operator reads for generic N (and written in the variable s , where D_s is the derivative d/ds)

$$\begin{aligned} \mathcal{V}_3 &= V_2 \cdot V_1, & V_1 &= D_s, \\ V_2 &= D_s^2 + \frac{1-5s^4}{s(1-s^4)} \cdot D_s + \frac{3s^6 - 7s^4 - 3s^2 - 1}{s^2 \cdot (1-s^4)^2} - \frac{4N(N+1)}{s^2}. \end{aligned} \quad (\text{F.4})$$

The form factors $C^{(2)}(N, N+1)$ expands as (with $x = w^2$)

$$\begin{aligned} C^{(2)}(N, N+1) &= x^{2N+3} \cdot \frac{2(2+N)^3(\Gamma(2N+3))^2}{(\Gamma(3+N))^4} \times \\ &\quad \left(1 + \frac{2(2N+3) \cdot (2N+5)^2}{(2+N)(3+N)} \cdot x + \frac{4(16N^3 + 148N^2 + 456N + 477)(2N+3)}{(3+N)(4+N)} \cdot x^2 \right. \\ &\quad \left. + \frac{8(2N+7)^2(16N^4 + 204N^3 + 956N^2 + 1983N + 1521)(2N+3)(2N+5)}{3(3+N)^2(2+N)(4+N)(5+N)} \cdot x^3 \right), \end{aligned} \quad (\text{F.5})$$

and are annihilated by an order six linear differential operator whose corresponding differential operator reads

$$\mathcal{V}_6 = V_3 \cdot V_2 \cdot V_1, \quad (\text{F.6})$$

where:

$$\begin{aligned} V_3 &= D_s^3 + \frac{4(1-5s^4)}{s(1-s^4)} \cdot D_s^2 \\ &\quad + \left(\frac{105s^8 - 16s^6 - 178s^4 - 16s^2 - 7}{s^2 \cdot (1-s^4)^2} - \frac{16N(N+1)}{s^2} \right) \cdot D_s \\ &\quad - \left(3 \frac{45s^{12} - 32s^{10} - 199s^8 - 96s^6 + 87s^4 + 3}{s^3 \cdot (1-s^4)^3} + \frac{48N(N+1)}{s^3} \right). \end{aligned} \quad (\text{F.7})$$

The first terms of $C^{(3)}(N, N+1)$ read (with $x = w^2$)

$$C^{(3)}(N, N+1) = 3072 \cdot \frac{(3+N)^2 64^N (\Gamma(N+5/2))^3 \cdot x^{6+3N}}{\pi^{3/2} (3+2N)^2 \cdot (\Gamma(N+4))^3} \times$$

$$\begin{aligned} & \left(1 + 6 \frac{(7+2N)^2(2+N) \cdot x}{(N+4)(3+N)} \right. \\ & \left. + 36 \frac{(5+2N)(4N^4+56N^3+287N^2+636N+507) \cdot x^2}{(3+N)(N+4)(N+5)} \right), \end{aligned} \quad (\text{F.8})$$

and are solution of an order-ten ODE whose corresponding linear differential operator factorizes as

$$\mathcal{V}_{10} = V_4 \cdot V_3 \cdot V_2 \cdot V_1, \quad (\text{F.9})$$

with

$$V_4 = D_s^4 + \frac{p_3}{p_4} \cdot D_s^3 + \frac{p_2}{p_4} \cdot D_s^2 + \frac{p_1}{p_4} \cdot D_s + \frac{p_0}{p_4}, \quad (\text{F.10})$$

where:

$$\begin{aligned} p_4 &= s^4 \cdot (1-s^4)^3 \cdot (1+s^2), \\ p_3 &= 10 \cdot s^3 \cdot (1-s^4)^2 \cdot (1+s^2) \cdot (1-5s^4), \\ p_2 &= -s^2 \cdot (1-s^4) \cdot (1+s^2) \cdot \left(40(1-s^4)^2 \cdot N \cdot (N+1) + 17 + 40s^2 \right. \\ &\quad \left. + 998s^4 + 40s^6 - 823s^8 \right), \\ p_1 &= s \cdot (1+s^2) \cdot \left(-8(1-s^4)^2(47-83s^4) \cdot N \cdot (N+1) - 175 - 72s^2 \right. \\ &\quad \left. - 3243s^4 + 2112s^6 + 16803s^8 + 968s^{10} - 5193s^{12} \right), \\ p_0 &= 144 \cdot (1-s^4)^3(1+s^2) \cdot N^4 + 288(1-s^4)^3(1+s^2) \cdot N^3 \\ &\quad - 144 \cdot (1-s^4)(1+s^2) \cdot \left(5 - 2s^2 - 50s^4 - 2s^6 + 17s^8 \right) \cdot N^2 \\ &\quad - 288 \cdot (1-s^4)(1+s^2) \cdot \left(3 - s^2 - 26s^4 - s^6 + 9s^8 \right) \cdot N \\ &\quad + 48s^2 \cdot \left(6 + 105s^2 + 63s^4 + 1705s^6 + 1247s^8 - 110s^{10} - 216s^{12} \right). \end{aligned} \quad (\text{F.11})$$

Remark: Unlike what we have seen for the diagonal $f_N^{(j)} = C^{(j)}(N, N)$, one notes that the linear differential equation \mathcal{V}_6 , which annihilates $C^{(2)}(N, N+1)$, solves $C^{(1)}(N, N+1)$ as well. Also, the linear differential equation \mathcal{V}_{10} which annihilates $C^{(1)}(N, N+1)$ and $C^{(3)}(N, N+1)$, solves $C^{(2)}(N, N+1)$ as well.

Appendix F.2. The linear differential equations in the scaling limit

The scaling limit is obtained by performing the variable change $s = 1 - y/N$, keeping the leading terms in N . However, since for the diagonal form factors the variable change was $t = 1 - x/N$ and since $t = s^4$, we will take, for easy comparison, the following variable change $s = 1 - x/(4N)$.

The scaling limit of \mathcal{V}_3 (corresponding to $C^{(1)}(N, N+1)$) has a *direct sum* factorization:

$$(V_2 \cdot V_1)^{scal} = L_1^{scal} \oplus L_2^{scal} \quad (\text{F.12})$$

Note the linear differential operators at the right-hand-side which are the operators given in Appendix B. L_2^{scal} is the scaling limit of the operators of the diagonal $f_N^{(1)}$.

The scaling limit of \mathcal{V}_6 (corresponding to $C^{(2)}(N, N+1)$) has *also a direct sum decomposition*:

$$(V_3 \cdot V_2 \cdot V_1)^{scal} = L_1^{scal} \oplus L_2^{scal} \oplus \tilde{L}_3^{scal}. \quad (\text{F.13})$$

Here also, \tilde{L}_3^{scal} is the operator appearing in Appendix B and corresponding to the scaling limit of the operator for $f_N^{(2)}$.

The scaling limit of \mathcal{V}_{10} (which corresponds to $C^{(1)}(N, N+1)$, $C^{(2)}(N, N+1)$, and $C^{(3)}(N, N+1)$) factorizes as:

$$(V_4 \cdot V_3 \cdot V_2 \cdot V_1)^{scal} = L_1^{scal} \oplus \tilde{L}_3^{scal} \oplus L_4^{scal} \cdot L_2^{scal}. \quad (\text{F.14})$$

Again, in the scaling limit, the linear differential operator corresponding to the diagonal $f_N^{(3)}$ appears.

Appendix G. Rational solutions of the symmetric (or exterior) square of L_n^{scal}

For n odd, the symmetric square of L_n^{scal} annihilate the following rational solutions r_n^{scal}

$$\begin{aligned} r_3^{scal} &= \frac{x^2 + 1}{x^2}, & r_5^{scal} &= \frac{x^6 + 4x^4 - 17x^2 - 9}{x^6}, \\ r_7^{scal} &= \frac{x^{14} + 9x^{12} - 105x^{10} + 1122x^8 + 2754x^6 - 15822x^4 - 4347x^2 + 2916}{x^{14}}, \\ x^{24} \cdot r_9^{scal} &= x^{24} + 16x^{22} - 354x^{20} + 9486x^{18} - 158364x^{16} - 1230840x^{14} \\ &\quad + 20545650x^{12} - 170513100x^{10} - 305967375x^8 + 552217500x^6 \\ &\quad - 1474858125x^4 - 603703125x^2 + 218700000. \end{aligned} \quad (\text{G.1})$$

For n even, the exterior square of L_n^{scal} annihilate the following rational solutions r_n^{scal}

$$\begin{aligned} r_4^{scal} &= \frac{x^2 - 2}{x^3}, & r_6^{scal} &= \frac{4x^8 - 24x^6 + 240x^4 + 144x^2 + 243}{x^9}, \\ x^{17} \cdot r_8^{scal} &= x^{16} - 12x^{14} + 300x^{12} - 5220x^{10} - 13734x^8 - 15804x^6 + 482760x^4 \\ &\quad + 90720x^2 + 327240, \\ x^{29} \cdot r_{10}^{scal} &= 8x^{28} - 160x^{26} + 7200x^{24} - 298080x^{22} + 7083540x^{20} + 52569000x^{18} \\ &\quad - 31590000x^{16} - 7017678000x^{14} + 113926753125x^{12} + 82387935000x^{10} \\ &\quad + 552374235000x^8 - 4206913200000x^6 - 1635103715625x^4 \\ &\quad - 1234903218750x^2 + 3690562500000. \end{aligned} \quad (\text{G.2})$$

For n odd, the symmetric square of the adjoint of the L_n^{scal} annihilate the following *polynomial* solutions R_n^{scal}

$$\begin{aligned} R_3^{scal} &= \frac{1}{x^2}, & R_5^{scal} &= x^6 - 12x^4 - 18x^2 - 18, \\ R_7^{scal} &= x^4 \cdot (2x^{18} - 360x^{16} + 15390x^{14} - 171450x^{12} + 251100x^{10} - 437400x^8 \\ &\quad - 2673000x^6 - 7290000x^4 - 9021375x^2 - 2916000), \\ R_9^{scal} &= x^{10} \cdot (4x^{36} - 3360x^{34} + 1010800x^{32} - 141699600x^{30} + 10025694000x^{28} \\ &\quad - 361966096800x^{26} + 6415388406000x^{24} - 50642114166000x^{22} \\ &\quad + 144901264095000x^{20} - 65747358180000x^{18} + 157049519760000x^{16} \\ &\quad + 1294768370700000x^{14} + 9174749528700000x^{12} + 49115481975000000x^{10} \\ &\quad + 178940697027750000x^8 + 406258455983625000x^6 + 504390999090234375x^4 \\ &\quad + 271021254918750000x^2 + 36456852600000000). \end{aligned} \quad (\text{G.3})$$

For n even, the exterior square of the *adjoint* of the L_n^{scal} annihilate the following *polynomial* solutions R_n^{scal}

$$\begin{aligned}
R_4^{scal} &= x \cdot (x^2 - 2), \quad R_6^{scal} = x^5 \cdot (x^{10} - 72x^8 + 792x^6 - 720x^4 - 1377x^2 - 1134), \\
R_8^{scal} &= x^{11} \cdot (4x^{24} - 1800x^{22} + 259200x^{20} - 15170400x^{18} + 370747125x^{16} \\
&\quad - 3461802300x^{14} + 8998897500x^{12} - 1729552500x^{10} + 6142736250x^8 \\
&\quad + 36349762500x^6 + 83751165000x^4 + 82668600000x^2 + 22471425000), \\
R_{10}^{scal} &= x^{19} \cdot (4x^{44} - 6272x^{42} + 3857280x^{40} - 1226991360x^{38} + 223629630000x^{36} \\
&\quad - 24414033420000x^{34} + 1620296469590400x^{32} - 64985230800000000x^{30} \\
&\quad + 1535802293434972500x^{28} - 20371435267457610000x^{26} \\
&\quad + 139451099666404050000x^{24} - 424167698945936610000x^{22} \\
&\quad + 409086586150282687500x^{20} - 24200532938759625000x^{18} \\
&\quad + 246154709199372750000x^{16} + 1920553025539034250000x^{14} \\
&\quad + 11369086319287068750000x^{12} + 49298397042819015000000x^{10} \\
&\quad + 143327291046534157500000x^8 + 257142668429453821875000x^6 \\
&\quad + 252273151762962774609375x^4 + 109262436886268613281250x^2 \\
&\quad + 12887098647274687500000).
\end{aligned} \tag{G.4}$$

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